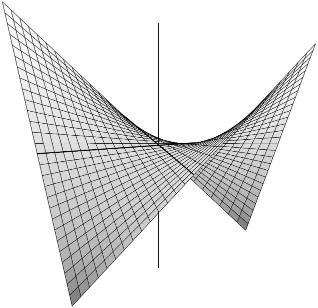
# Linear and Bilinear Algebra I



**DR Lama Tarsissi**
Sorbonne University of Abu Dhabi
MATH222

# Contacts

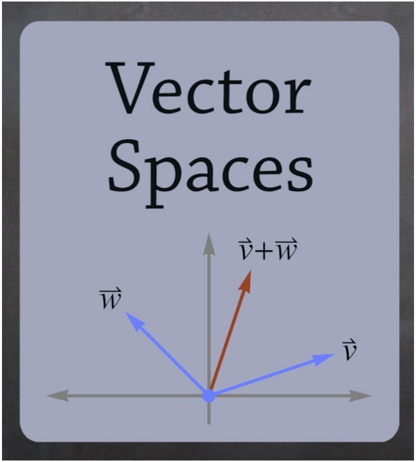
## Course Leader

Lama Tarsissi (lama.tarsissi@sorbonne.ae)

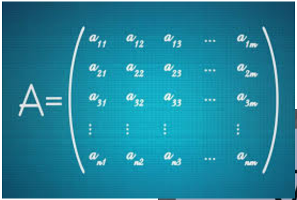
## Course follow up

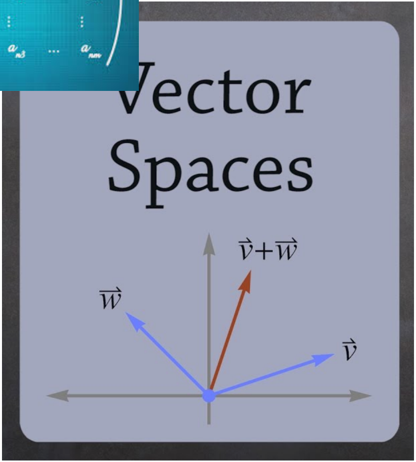
1-Blackboard
2-lamatarsissi.com

# you need to have

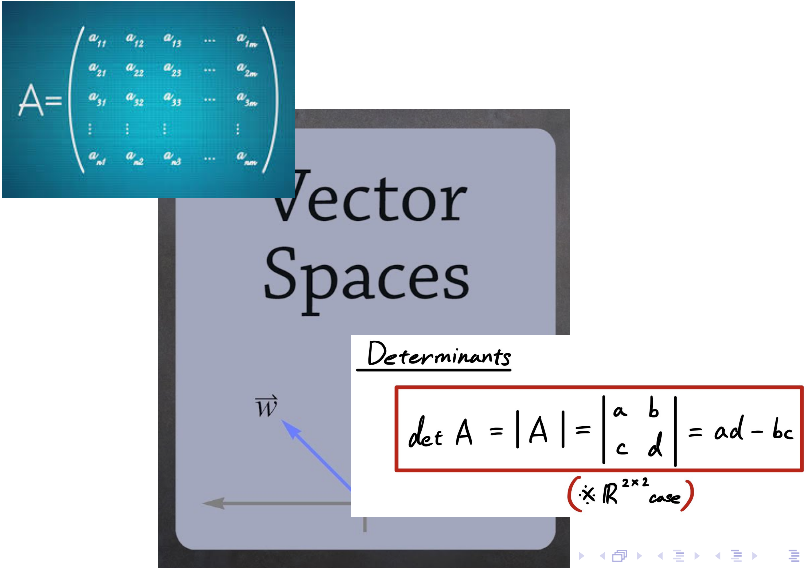


# you need to have

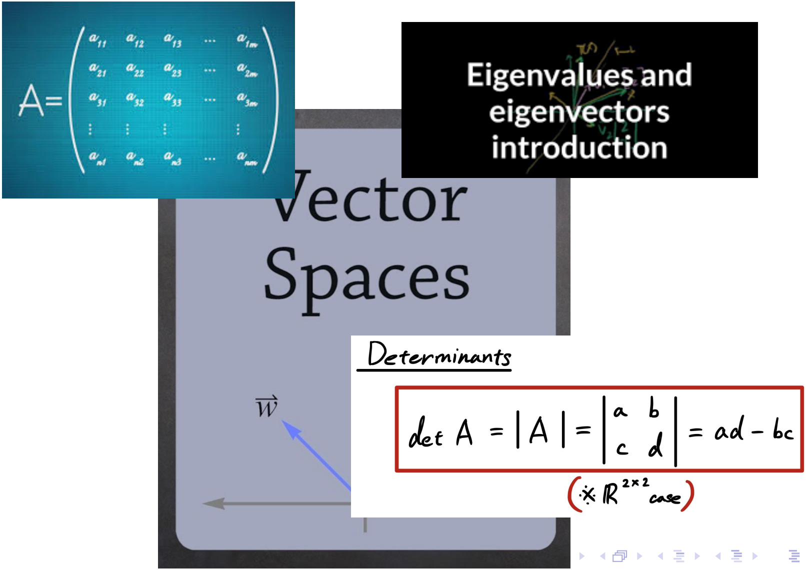


# you need to have

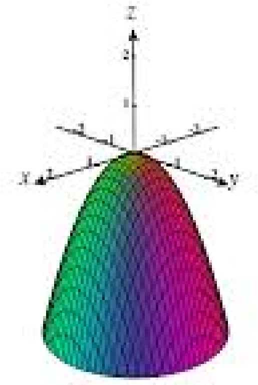


# you need to have



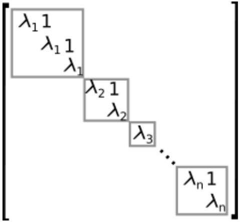
# To aim:

**Quadratic forms**



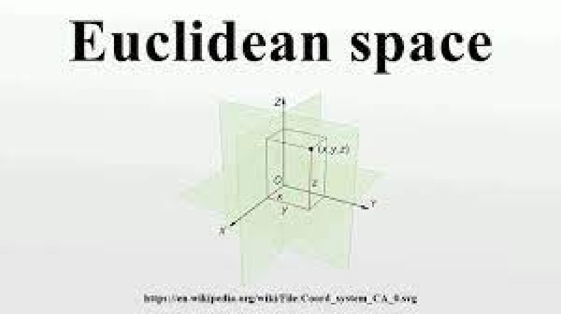
# To aim:

**Diagonalization of linear applications**



# To aim:

**Euclidean forms**



# Using three diffrent point of views

*60 hours*

1. The algorithms that allow an effective calculation of the studied objects

# Using three diffrent point of views

60 *hours*

1. The algorithms that allow an effective calculation of the studied objects
2. The geometrical point of view which makes it possible to visualize them,

# Using three diffrent point of views

60 *hours*

1. The algorithms that allow an effective calculation of the studied objects
2. The geometrical point of view which makes it possible to visualize them,
3. The Algebra which gives an abstract representation simplifying the study.

# Using three diffrent point of views

60 *hours*

1. The algorithms that allow an effective calculation of the studied objects
2. The geometrical point of view which makes it possible to visualize them,
3. The Algebra which gives an abstract representation simplifying the study.

# EU learning objectives

## Direct Sums, Characteristic Polynomials and Duality:

- Recall of direct sums, determinants....
- Dual spaces, Kernal and images of dual maps.....

## Pre-Hilbert Real Vector Spaces

- Euclidean spaces
- Gram-Schmidt's algorithm
- Orthogonal projection and minimization problems

*Midterm I*

## Symmetric Bilinear Forms on Euclidean spaces

- Symmetric bilinear forms and quadratic forms.
- Orthogonality with respect to a quadratic form and signature.
- Sylvester's law of inertia and Gauss' algorithm.

## Pre-Hilbert Complex Vector Spaces

- Hilbertian inner product
- Cauchy-Schwarz inequality.
- Hilbert norms

*Midterm II*

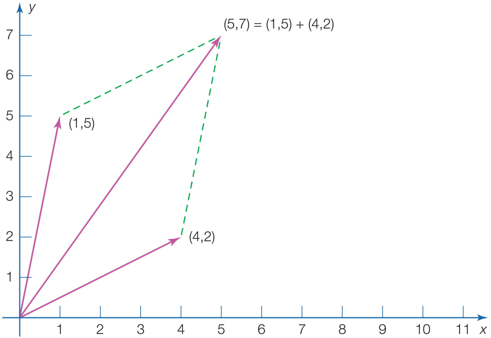
# Ch1. RECALL

# Vectors

A **vector** $u$ in the vector space $\mathbb{R}^n$ is an ordered set of $n$ real numbers: $u = (u_1, u_2, \ldots, u_n)$. The real number $a_k$ is called the $k^{th}$ component or coordinate of $u$.

# Vectors

A **vector** $u$ in the vector space $\mathbb{R}^n$ is an ordered set of $n$ real numbers: $u = (u_1, u_2, \ldots, u_n)$. The real number $a_k$ is called the $k^{th}$ component or coordinate of $u$.



# Some Properties over the vectors

- Column vector: $u = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$

# Some Properties over the vectors

- Column vector: $u = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$
- The **zero vector** $0 = (0, 0, \dots, 0)$.

# Some Properties over the vectors

- Column vector: $u = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$
- The **zero vector** $0 = (0, 0, \dots, 0)$.
- $-u = (-u_1, -u_2, \dots, -u_n)$;

# Some Properties over the vectors

- Column vector: $u = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$
- The **zero vector** $0 = (0, 0, \dots, 0)$.
- $-u = (-u_1, -u_2, \dots, -u_n)$; $-(-u) = u$

# Some Properties over the vectors

- Column vector: $u = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$
- The **zero vector** $0 = (0, 0, \dots, 0)$.
- $-u = (-u_1, -u_2, \dots, -u_n)$; $-(-u) = u$
- $u + v = (u_1, u_2, \dots, u_n) + = (v_1, v_2, \dots, v_n) =$

$$(u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

- for $k \in \mathbb{R}$ and $u = (u_1, u_2, \ldots, u_n)$, the *scalar product* $ku$ is given by:

$$ku \equiv (ku_1, ku_2, \ldots, ku_n)$$

- for $k \in \mathbb{R}$ and $u = (u_1, u_2, \ldots, u_n)$, the *scalar product* $ku$ is given by:

$$ku \equiv (ku_1, ku_2, \ldots, ku_n)$$

- The *Dot( inner) product* between u and vis given by:

$$u.v = \sum_{k=1}^{n} u_k v_k$$

- for $k \in \mathbb{R}$ and $u = (u_1, u_2, \ldots, u_n)$, the *scalar product* $ku$ is given by:

$$ku \equiv (ku_1, ku_2, \ldots, ku_n)$$

- The *Dot( inner) product* between u and vis given by:

$$u.v = \sum_{k=1}^{n} u_k v_k$$

- *The norm(length)* in $\mathbb{R}^n$ of $u = (u_1, u_2, \ldots, u_n)$ is given by:

$$||u|| = \sqrt{u.u} = \sqrt{u_1^2 + u_2^2 \ldots u_n^2}$$

# Examples

Let $u = (3, -5, 2, 1)$ and $v = (4, 1, -2, 5)$ be two real vectors in $\mathbb{R}^4$.
Compute: $-u, u + v, u.v, ||v||$ and $-4v$.

# Examples

Let $u = (3, -5, 2, 1)$ and $v = (4, 1, -2, 5)$ be two real vectors in $\mathbb{R}^4$.
Compute: $-u, u + v, u.v, ||v||$ and $-4v$.

## Solution

- $-u = (-3, 5, -2, -1)$
- $u + v = (7, -4, 0, 6)$
- $u.v = 3.4 + (-5).1 + 2.(-2) + 1.5 = 8$
- $||v|| = \sqrt{16 + 1 + 4 + 25} = \sqrt{46}$
- $-4v = (-16, -4, 8, -20)$

# Matrix

An $m \times n$ matrix: the m rows are horizontal and the n columns are vertical. Each element of a matrix is often denoted by a variable with two subscripts. For example, $a_{21}$ represents the element at the **second row** and **first column** of the matrix. $A = \{a_{i,j}\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & \dots & n \end{array} \\
\begin{bmatrix}
a_{11} & a_{12} & \dots & a_{1n} \\
a_{21} & a_{22} & \dots & a_{2n} \\
a_{31} & a_{32} & \dots & a_{3n} \\
\vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn}
\end{bmatrix}
\end{array}
$$

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

| Name | Size | Example |
|---|---|---|
| Row vector | $1 \times n$ | $\begin{bmatrix} 3 & 7 & 2 \end{bmatrix}$ |
| Column vector | $n \times 1$ | $\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ |
| Square matrix | $n \times n$ | $\begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 2 & 6 & 3 \end{bmatrix}$ |

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

- If all entries outside the main diagonal are zero, A is called a diagonal matrix.

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

- If all entries outside the main diagonal are zero, A is called a diagonal matrix.
- if all entries of A above the main diagonal are zero, A is called a lower triangular matrix.

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

- If all entries outside the main diagonal are zero, A is called a diagonal matrix.
- if all entries of A above the main diagonal are zero, A is called a lower triangular matrix.
- If all entries of A below the main diagonal are zero, A is called an upper triangular matrix.

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

- If all entries outside the main diagonal are zero, A is called a diagonal matrix.
- if all entries of A above the main diagonal are zero, A is called a lower triangular matrix.
- If all entries of A below the main diagonal are zero, A is called an upper triangular matrix.

| Name | Example with $n = 3$ |
|---|---|
| Diagonal matrix | $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ |
| Lower triangular matrix | $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ |
| Upper triangular matrix | $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ |

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

- If all entries outside the main diagonal are zero, A is called a diagonal matrix.
- if all entries of A above the main diagonal are zero, A is called a lower triangular matrix.
- If all entries of A below the main diagonal are zero, A is called an upper triangular matrix.
- The identity matrix $I_n$ of size n is the $n \times n$ matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0.

# Some properties on Matrices

- A row vector of length $n$ is a $1 \times n$ matrix.
- A column vector of length $m$ is a $m \times 1$ matrix.
- A square matrix of length $n^2$ is a $n \times n$ matrix.

- If all entries outside the main diagonal are zero, A is called a diagonal matrix.
- if all entries of A above the main diagonal are zero, A is called a lower triangular matrix.
- If all entries of A below the main diagonal are zero, A is called an upper triangular matrix.
- The identity matrix $I_n$ of size n is the $n \times n$ matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0.

$$\mathbf{I}_1 = [1], \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ldots, \mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

# Addition and Scalar multiplication

## Addition

The sum $A + B$ of two $m \times n$ matrices A and B is calculated entrywise:

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}, \ \textit{where } 1 \leq i \leq m \textit{ and } 1 \leq j \leq n.$$

# Addition and Scalar multiplication

## Addition

The sum $A + B$ of two $m \times n$ matrices A and B is calculated entrywise:

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}, \text{ where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

# Addition and Scalar multiplication

## Addition

The sum $A + B$ of two $m \times n$ matrices A and B is calculated entrywise:

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}, \text{ where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

## Scalar multiplication

The product $c$A of a number c (also called a scalar) and a matrix $A$ is computed by multiplying every entry of A by c:

$$(cA)_{i,j} = cA_{i,j}.$$

# Addition and Scalar multiplication

## Addition

The sum $A + B$ of two $m \times n$ matrices A and B is calculated entrywise:

$$(A + B)_{i,j} = A_{i,j} + B_{i,j}, \text{ where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

## Scalar multiplication

The product $cA$ of a number c (also called a scalar) and a matrix $A$ is computed by multiplying every entry of A by c:

$$(cA)_{i,j} = cA_{i,j}.$$

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot -3 \\ 2 \cdot 4 & 2 \cdot -2 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

# Multiplication

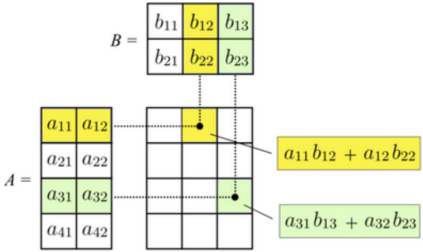
Multiplication of two matrices is defined if and only if the **number of columns** of the left matrix is **the same** as the **number of rows** of the right matrix where $AB \neq BA$.
If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their matrix product AB is the $m \times p$ matrix whose entries are given by dot product of the corresponding row of A and the corresponding column of B as follows:

$$[AB]_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \dots a_{i,n}b_{n,j} = \sum_{r=1}^{n} a_{i,r}b_{r,j}.$$

# Multiplication

Multiplication of two matrices is defined if and only if the **number of columns** of the left matrix is **the same** as the **number of rows** of the right matrix where $AB \neq BA$.
If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their matrix product AB is the $m \times p$ matrix whose entries are given by dot product of the corresponding row of A and the corresponding column of B as follows:

$$[AB]_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \dots a_{i,n}b_{n,j} = \sum_{r=1}^{n} a_{i,r}b_{r,j}.$$



# Transposition and submatrix

## Transposition

The transpose of a $m \times n$ matrix A is the $n \times m$ matrix $A^T$ (also denoted $A^{tr}$ or $t_A$) formed by turning rows into columns and vice versa:

$$(A^T)_{i,j} = A_{j,i}.$$

# Transposition and submatrix

## Transposition

The transpose of a $m \times n$ matrix A is the $n \times m$ matrix $A^T$ (also denoted $A^{tr}$ or $t_A$) formed by turning rows into columns and vice versa:

$$(A^T)_{i,j} = A_{j,i}.$$

## Submatrix

A submatrix of a matrix is obtained by deleting any collection of rows and/or columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & \color{red}{2} & 3 & 4 \\ 5 & \color{red}{6} & 7 & 8 \\ \color{red}{9} & \color{red}{10} & \color{red}{11} & \color{red}{12} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \end{bmatrix}.$$

# Some theorems

Let $A, B$, and $C$ be three $m \times n$ matices and $k$ a scalar in $\mathbb{R}$

1. $(AB)C = A(BC)$ Associative law
2. $A(B + C) = AB + AC$ left distributive law
3. $(B + C)A = BA + CA$ right distributive law
4. $AB \neq BA$ NOT Commutative
5. $k(AB) = (kA)B = A(kB)$
6. $(A^T)^T = A$
7. $(AB)^T = B^T A^T$

# Some generalities

1. The set of all the square matrices of dimension $n$ is denoted $M_n(\mathbb{R})$.

# Some generalities

1. The set of all the square matrices of dimension $n$ is denoted $M_n(\mathbb{R})$.
2. For each matrix of dimension $m \times n$ we can associate a linear map defined from $\mathbb{R}^n$ into $\mathbb{R}^m$ which relates the vector $x = (x_1, \dots, x_n)$ to the vector $y = (y_1, \dots, y_m)$ with the following formula:

$$y_i = \sum_{j=1}^{n} a_{i,j} x_j, \forall 1 \leq i \leq m.$$

# Some generalities

1. The set of all the square matrices of dimension $n$ is denoted $M_n(\mathbb{R})$.
2. For each matrix of dimension $m \times n$ we can associate a linear map defined from $\mathbb{R}^n$ into $\mathbb{R}^m$ which relates the vector $x = (x_1, \dots, x_n)$ to the vector $y = (y_1, \dots, y_m)$ with the following formula:

$$y_i = \sum_{j=1}^{n} a_{i,j} x_j, \forall 1 \leq i \leq m.$$

3. We use the notation $y = Ax$, $y$ and $x$ are column vectors.

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For each matrix $\{a_{i,j}\}$, it is associated a system of linear equations:

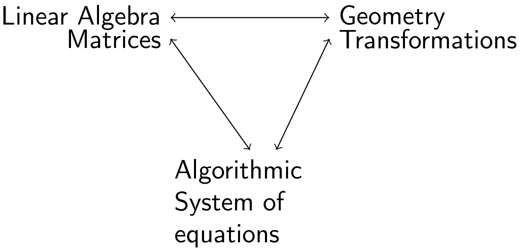
For each matrix $\{a_{i,j}\}$, it is associated a system of linear equations:

$$\begin{cases} a_{1,1}x_1 & + & \dots & + & a_{1,n}x_n & = & y_1 \\ \vdots & & & & \vdots & & \vdots \\ a_{m,1}x_1 & + & \dots & + & a_{m,n}x_n & = & y_m \end{cases}$$

For each matrix $\{a_{i,j}\}$, it is associated a system of linear equations:

$$\begin{cases} a_{1,1}x_1 & + & \dots & + & a_{1,n}x_n & = & y_1 \\ \vdots & & & & \vdots & & \vdots \\ a_{m,1}x_1 & + & \dots & + & a_{m,n}x_n & = & y_m \end{cases}$$

This system of linear equations express the coordinates of the vector $y$ in function of the one of $x$.




- In Linear Algebra, we add and multiply matrices,
- In Geometry, we compose transformations,
- In Algoritmic, we manipulate system ofequations.

# Dependent Columns

The columns of $A$ are "dependent" if one column is a **combination** of the other columns.
Another way to describe dependence: $Ax = 0$ for some vector $x$ (other than $x = 0$)

# Dependent Columns

The columns of A are "dependent" if one column is a **combination** of the other columns.
Another way to describe dependence: $Ax = 0$ for some vector x (other than $x = 0$)

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \text{ have } \textbf{dependent columns}$$

# Dependent Columns

The columns of A are "dependent" if one column is a **combination** of the other columns.
Another way to describe dependence: $Ax = 0$ for some vector $x$ (other than $x = 0$)

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \text{ have } \textbf{dependent columns}$$

$$\text{Reasons : Column 2 of } A_1 = 2\,(\text{Column 1}) \qquad A_2 \text{ times } \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ gives } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A_3$ has 3 columns in 2-dimensional space. Three vectors in a plane : **Dependent !**

# Independent Columns

The columns of A are "independent" if **no column** is a **combination** of the other columns.
Another way to say it: $Ax = 0$ only when $x = 0$.

# Independent Columns

The columns of A are "independent" if **no column** is a **combination** of the other columns.
Another way to say it: Ax = 0 only when x = 0.

$$A_4 = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \text{ and } A_5 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_6 = I \text{ have independent columns}$$

# Independent Columns

The columns of A are "independent" if **no column** is a **combination** of the other columns.
Another way to say it: Ax = 0 only when x = 0.

$$A_4 = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \text{ and } A_5 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_6 = I \text{ have independent columns}$$

What about the rows of A1 to A6 ? A1, A2, A4 have dependent rows. Possibly also A3?

# Independent Columns

The columns of A are "independent" if **no column** is a **combination** of the other columns.
Another way to say it: Ax = 0 only when x = 0.

$$A_4 = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \text{ and } A_5 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_6 = I \text{ have independent columns}$$

What about the rows of A1 to A6 ? A1, A2, A4 have dependent rows. Possibly also A3?

## N.B

**For any square matrix: Columns are independent if and only if rows are independent.**
Number of independent rows = Number of independent columns

# Rank of a matrix

## Definition

The Kernel of a matrix $A$ of dimension $m \times n$ is composed of vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

$$ker(A) = \{x \in \mathbb{R}^n | Ax = 0\}.$$

# Rank of a matrix

## Definition

The Kernel of a matrix $A$ of dimension $m \times n$ is composed of vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

$$ker(A) = \{x \in \mathbb{R}^n | Ax = 0\}.$$

## Definition

The image of $A$ is composed of vectors in $\mathbb{R}^m$ of the form $Ax$; $x \in \mathbb{R}^n$.

$$Im(A) = \{Ax \in \mathbb{R}^m | x \in \mathbb{R}^n\}.$$

# Rank of a matrix

### Definition

The Kernel of a matrix $A$ of dimension $m \times n$ is composed of vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

$$ker(A) = \{x \in \mathbb{R}^n | Ax = 0\}.$$

### Definition

The image of $A$ is composed of vectors in $\mathbb{R}^m$ of the form $Ax$; $x \in \mathbb{R}^n$.

$$Im(A) = \{Ax \in \mathbb{R}^m | x \in \mathbb{R}^n\}.$$

$$rank(A) = dim\ (Im(A)).$$

# Some properties

1. The image of a matrix is equal to the vector space generated by its columns.

# Some properties

1. The image of a matrix is equal to the vector space generated by its columns.
2. The rank is equal to the dimension of this space.

# Some properties

1. The image of a matrix is equal to the vector space generated by its columns.
2. The rank is equal to the dimension of this space.
3. The kernel and the image of a matrix are vector spaces.

# Some properties

1. The image of a matrix is equal to the vector space generated by its columns.
2. The rank is equal to the dimension of this space.
3. The kernel and the image of a matrix are vector spaces.
4. The row of a matrix is an integer which is zero if and only if all the coefficients of the matrix are zero.

# Some properties

1. The image of a matrix is equal to the vector space generated by its columns.
2. The rank is equal to the dimension of this space.
3. The kernel and the image of a matrix are vector spaces.
4. The row of a matrix is an integer which is zero if and only if all the coefficients of the matrix are zero.

**Proposition (Rank formula)**

*Let $A$ be a matrix of dimension $m \times n$,*

$$rank(A) + dim(ker(A)) = n.$$

# Matrix in an Echelon form

## Definition

A matrix is called in a row echelon form if the number of consecutif zeros at the begining of each row increases strictly from a row to another, untill having nothing but zeros.

# Matrix in an Echelon form

**Definition**

A matrix is called in a row echelon form if the number of consecutif zeros at the begining of each row increases strictly from a row to another, untill having nothing but zeros.

We call *pivot*(leading), the first non zero number in each row.

# Matrix in an Echelon form

**Definition**

A matrix is called in a row echelon form if the number of consecutif zeros at the begining of each row increases strictly from a row to another, untill having nothing but zeros.

We call *pivot*(leading), the first non zero number in each row.

$$\begin{pmatrix} 0 & \mathbf{3} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{1} & 5 \end{pmatrix}, \begin{pmatrix} \mathbf{2} & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 2 & 0 \\ 0 & 0 & \mathbf{2} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 4 & 0 & 0 & 5 \\ 0 & 0 & \mathbf{1} & 1 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 3 \end{pmatrix}.$$

# Matrix in an Echelon form

**Definition**

A matrix is called in a row echelon form if the number of consecutif zeros at the begining of each row increases strictly from a row to another, untill having nothing but zeros.

We call *pivot*(leading), the first non zero number in each row.

$$\begin{pmatrix} 0 & \mathbf{3} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{1} & 5 \end{pmatrix}, \begin{pmatrix} \mathbf{2} & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 2 & 0 \\ 0 & 0 & \mathbf{2} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 4 & 0 & 0 & 5 \\ 0 & 0 & \mathbf{1} & 1 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 3 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 5 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

# Matrix in an Echelon form

**Definition**

A matrix is called in a row echelon form if the number of consecutif zeros at the begining of each row increases strictly from a row to another, untill having nothing but zeros.

We call *pivot*(leading), the first non zero number in each row.

$$\begin{pmatrix} 0 & \mathbf{3} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{1} & 5 \end{pmatrix}, \begin{pmatrix} \mathbf{2} & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 2 & 0 \\ 0 & 0 & \mathbf{2} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 4 & 0 & 0 & 5 \\ 0 & 0 & \mathbf{1} & 1 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 3 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 5 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The square echelon matrices are triangular: $a_{i,j} = 0, \ \textit{if} \ i > j$.

# Reduced Echelon form

## Definition

An echelon matrix is in a reduced echelon form if:

- The pivots are all equal to 1
- In a column that contains a pivot, all the other elements are null but the pivot itself.

# Reduced Echelon form

**Definition**

An echelon matrix is in a reduced echelon form if:

- The pivots are all equal to 1
- In a column that contains a pivot, all the other elements are null but the pivot itself.

$$\begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 0 & 5 \\ 0 & \mathbf{1} & 2 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 3 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 4 & 0 & 0 & 5 \\ 0 & 0 & \mathbf{1} & 0 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 3 \end{pmatrix}.$$

# Reduced Echelon form

**Definition**

An echelon matrix is in a reduced echelon form if:

- The pivots are all equal to 1
- In a column that contains a pivot, all the other elements are null but the pivot itself.

$$\begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 0 & 5 \\ 0 & \mathbf{1} & 2 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 3 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 4 & 0 & 0 & 5 \\ 0 & 0 & \mathbf{1} & 0 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 3 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Rank of a matrix in an echelon form

**Proposition**

*The rank of an echelon form matrix is equal to its number of pivots.It is also equal to the number of non-identically zero rows. The dimension of the kernel of a scaled matrix is equal to the number of columns that do not contain pivots.*

# Rank of a matrix in an echelon form

**Proposition**

*The rank of an echelon form matrix is equal to its number of pivots.It is also equal to the number of non-identically zero rows. The dimension of the kernel of a scaled matrix is equal to the number of columns that do not contain pivots.*

It is easy to solve a system of linear equations that is associated to reduced echelon form matrices.

# Method I

Let A be a reduced echelon form matrix. We are looking to solve the system described earlier: Given the vector $y$, we are looking for all the $x$ for which $Ax = y$.
The variables $x_1, \ldots, x_n$ situated at the position of pivots are called *principal variables.*

- We introduce a parameter for each variable which is not principal.
- We express the principal variables in function of these parameters

We get in this case a parametric equation for the set of solutions for the system.

# Example

Let $y_1, y_2$ be two given real numbers, let us find the solutions $(x_1, x_2, x_3) \in \mathbb{R}^3$ for the following system:

$$\begin{cases} x_1 & + & & + & 3x_3 & = & y_1 \\ & & x_2 & + & x_3 & = & y_2 \end{cases}$$

associated to the reduce echelon matrix form: $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

# Solution

- The variables $x_1, x_2$ are principal variables, so we introduce the parameter $\lambda$ and we set $x_3 = \lambda$.
- We express the principal variables $x_1$ and $x_2$ in function of $\lambda$,in order to get the parametric equation:

$$\begin{cases} x_1 = y_1 - 3x_3 = y_1 - 3\lambda \\ x_2 = y_2 - x_3 = y_2 - \lambda \end{cases} \Rightarrow \begin{cases} x_1 = y_1 - 3\lambda \\ x_2 = y_2 - \lambda \\ x_3 = \lambda \end{cases}$$

- The vectorial notation will be: $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}.$$

# Conclusions

- For all valuesof $y_1$ and $y_2$, the system has an infinity of solutions.

# Conclusions

- For all valuesof $y_1$ and $y_2$, the system has an infinity of solutions.
- The set of solutions is the straight line directed by the vector $(-3, -1, 1)$ and passing by the fixed point of coordinates $(y_1, y_2, 0)$.

# Conclusions

- For all valuesof $y_1$ and $y_2$, the system has an infinity of solutions.
- The set of solutions is the straight line directed by the vector $(-3, -1, 1)$ and passing by the fixed point of coordinates $(y_1, y_2, 0)$.
- We can get the kernel of A byletting $y_1 = y_2 = 0$:

$$ker(A) = \{\lambda(-3, -1, 1) | \lambda \in \mathbb{R}\} = vect\left( \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right)$$

# Conclusions

- For all valuesof $y_1$ and $y_2$, the system has an infinity of solutions.
- The set of solutions is the straight line directed by the vector $(-3, -1, 1)$ and passing by the fixed point of coordinates $(y_1, y_2, 0)$.
- We can get the kernel of A byletting $y_1 = y_2 = 0$:

$$ker(A) = \{\lambda(-3, -1, 1) | \lambda \in \mathbb{R}\} = vect\left( \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right)$$

- The dimension of the kernel is equal to 1, which is the numberof columns in A without leadings; i.e the third one.

# Method II:Gaussian algorithm

The Gaussian elimination, also called Row elimination , is an algorithm for solving systems of linear equations. There are three types of row operations that are allowed:

1. Swapping two rows, interchange the $i^{th}$ row and the $j^{th}$ row: $R_i \longleftrightarrow R_j$
2. Multiply the $i^{th}$ row by a nonzero scalar $k$:$R_i \to kR_i, k \neq 0$
3. Replace the $i^{th}$ row by $k$ times the $j^{th}$ row plus the $i^{th}$ row: $R_i \to kR_j + R_i$.

# Method II:Gaussian algorithm

The Gaussian elimination, also called Row elimination , is an algorithm for solving systems of linear equations. There are three types of row operations that are allowed:

1. Swapping two rows, interchange the $i^{th}$ row and the $j^{th}$ row: $R_i \longleftrightarrow R_j$
2. Multiply the $i^{th}$ row by a nonzero scalar $k$:$R_i \to kR_i, k \neq 0$
3. Replace the $i^{th}$ row by $k$ times the $j^{th}$ row plus the $i^{th}$ row: $R_i \to kR_j + R_i$.
   This algorihm allows us to get the associated matrix to the echelon form.

# Gaussian elimination to reduced form

In order to get the reduced echelon form of the matrix, we need to continue the algorithm by applying the following steps:

1. Reduce the matrix to an echelon form; denote the leading nonzero entries $a_{1j_1}, a_{2j_2}, \ldots, a_{rj_r}$.
2. If $a_{rj_r} \neq 1$, mutiply the last nonzero row, $R_r$ by $1/a_{rj_r}$; Then use $a_{rj_r} = 1$ as a pivot to obtain $0's$ above the pivot. Repeat the process with $R_{r-1}, R_{r-2}, \ldots, R_2$. Finally, if necessary, multiply $R_1$ by $1/a_{1j_1}$ to make $a_{1j_1} = 1$.

# Gaussian elimination to reduced form

In order to get the reduced echelon form of the matrix, we need to continue the algorithm by applying the following steps:

1. Reduce the matrix to an echelon form; denote the leading nonzero entries $a_{1j_1}, a_{2j_2}, \dots, a_{rj_r}$.
2. If $a_{rj_r} \neq 1$, mutiply the last nonzero row, $R_r$ by $1/a_{rj_r}$; Then use $a_{rj_r} = 1$ as a pivot to obtain $0's$ above the pivot. Repeat the process with $R_{r-1}, R_{r-2}, \dots, R_2$. Finally, if necessary, multiply $R_1$ by $1/a_{1j_1}$ to make $a_{1j_1} = 1$.
   The matrix now is in a reduced echelon form. Step 2 is sometimes called back-substitution.

# Example on the Gaussian elemination

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 12 \\ 1 & 2 & 5 \end{pmatrix}$

# Example on the Gaussian elemination

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 12 \\ 1 & 2 & 5 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 6 & 6 \\ 0 & 3 & 3 \end{pmatrix} \quad \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array}$$

# Example on the Gaussian elemination

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 12 \\ 1 & 2 & 5 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 6 & 6 \\ 0 & 3 & 3 \end{pmatrix} \quad \begin{matrix} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - L_1 \end{matrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} L_3 \leftarrow L_3 - \frac{3}{2}L_2 \\ L_4 \leftarrow L_4 - \frac{3}{4}L_2 \end{matrix}$$

# Example on the Gaussian elemination

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 12 \\ 1 & 2 & 5 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 6 & 6 \\ 0 & 3 & 3 \end{pmatrix} \quad \begin{matrix} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - L_1 \end{matrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} L_3 \leftarrow L_3 - \frac{3}{2}L_2 \\ L_4 \leftarrow L_4 - \frac{3}{4}L_2 \end{matrix}$$

Till here, we get the echelon form of $A$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} L_3 \leftarrow L_3 - \frac{3}{2}L_2 \\ L_4 \leftarrow L_4 - \frac{3}{4}L_2 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} L_3 \leftarrow L_3 - \frac{3}{2}L_2 \\ L_4 \leftarrow L_4 - \frac{3}{4}L_2 \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_2 \leftarrow \frac{1}{4}L_2$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} L_3 \leftarrow L_3 - \frac{3}{2}L_2 \\ L_4 \leftarrow L_4 - \frac{3}{4}L_2 \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_2 \leftarrow \frac{1}{4}L_2$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_1 \leftarrow L_1 + L_2$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} L_3 \leftarrow L_3 - \frac{3}{2}L_2 \\ L_4 \leftarrow L_4 - \frac{3}{4}L_2 \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_2 \leftarrow \frac{1}{4}L_2$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_1 \leftarrow L_1 + L_2$$

Here, A is in a reduced echelon form.

# Augmented matrix

This method is used to find the solution for a system of linear equations of the form $AX = B$.

This matrix is obtained by appending the columns of the two given matrices, A and B, usually for the purpose of performing the same elementary row operations on each of the given matrices.

# Example

| System of equations | Row operations | Augmented matrix |
|---|---|---|
| $\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$ | | $\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$ |
| $\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$ | $\begin{aligned} L_2 + \frac{3}{2}L_1 &\to L_2 \\ L_3 + L_1 &\to L_3 \end{aligned}$ | $\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$ |
| $\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$ | $L_3 + -4L_2 \to L_3$ | $\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$ |
| The matrix is now in echelon form (also called triangular form) | | |

# Example

| | | |
|---|---|---|
| $\begin{aligned} 2x + y \quad &= 7 \\ \tfrac{1}{2}y \quad &= \tfrac{3}{2} \\ -z &= 1 \end{aligned}$ | $\begin{aligned} L_2 + \tfrac{1}{2}L_3 &\to L_2 \\ L_1 - L_3 &\to L_1 \end{aligned}$ | $\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right]$ |
| $\begin{aligned} 2x + y \quad &= 7 \\ y \quad &= 3 \\ z &= -1 \end{aligned}$ | $\begin{aligned} 2L_2 &\to L_2 \\ -L_3 &\to L_3 \end{aligned}$ | $\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$ |
| $\begin{aligned} x \quad &= 2 \\ y \quad &= 3 \\ z &= -1 \end{aligned}$ | $\begin{aligned} L_1 - L_2 &\to L_1 \\ \tfrac{1}{2}L_1 &\to L_1 \end{aligned}$ | $\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$ |

# Conlusion

We put a matrix in an echelon form for the following reasons:

- Calculate the *rank* of a matrix.
  - It is equal to the number of pivots in its echelon form.

# Conlusion

We put a matrix in an echelon form for the following reasons:

- Calculate the *rank* of a matrix.
  - It is equal to the number of pivots in its echelon form.
- Calculate the $\dim(\ker(A))$.
  - It is equal to the number of columns in its echelon form without pivots.

# Conlusion

We put a matrix in an echelon form for the following reasons:

- Calculate the *rank* of a matrix.
  - It is equal to the number of pivots in its echelon form.
- Calculate the dim(ker(A)).
  - It is equal to the number of columns in its echelon form without pivots.
- Calculate the ker of the matrix.
  - It is equal to the kernel of its echelon form.

# Conlusion

We put a matrix in an echelon form for the following reasons:

- Calculate the *rank* of a matrix.
  - It is equal to the number of pivots in its echelon form.
- Calculate the dim(ker(A)).
  - It is equal to the number of columns in its echelon form without pivots.
- Calculate the ker of the matrix.
  - It is equal to the kernel of its echelon form.
- Calculate the basis of the image of a matrix.
  - Take the columns of the associated matrix on the pivots position for the reduced form.

# Conlusion

We put a matrix in an echelon form for the following reasons:

- Calculate the *rank* of a matrix.
  - It is equal to the number of pivots in its echelon form.
- Calculate the dim(ker(A)).
  - It is equal to the number of columns in its echelon form without pivots.
- Calculate the ker of the matrix.
  - It is equal to the kernel of its echelon form.
- Calculate the basis of the image of a matrix.
  - Take the columns of the associated matrix on the pivots position for the reduced form.
- Determine if a family of vectors in $\mathbb{R}^n$ is free.
  - The rank of the family must be equal to the number of vectors.

# Conlusion

We put a matrix in an echelon form for the following reasons:

- Calculate the *rank* of a matrix.
  - It is equal to the number of pivots in its echelon form.
- Calculate the dim(ker(A)).
  - It is equal to the number of columns in its echelon form without pivots.
- Calculate the ker of the matrix.
  - It is equal to the kernel of its echelon form.
- Calculate the basis of the image of a matrix.
  - Take the columns of the associated matrix on the pivots position for the reduced form.
- Determine if a family of vectors in $\mathbb{R}^n$ is free.
  - The rank of the family must be equal to the number of vectors.
- Solve a system of linear equations with a second member
  - Put the system in a reduced echelon form.

# Conlusion

We put a matrix in an echelon form for the following reasons:

- Calculate the *rank* of a matrix.
  - It is equal to the number of pivots in its echelon form.
- Calculate the dim(ker(A)).
  - It is equal to the number of columns in its echelon form without pivots.
- Calculate the ker of the matrix.
  - It is equal to the kernel of its echelon form.
- Calculate the basis of the image of a matrix.
  - Take the columns of the associated matrix on the pivots position for the reduced form.
- Determine if a family of vectors in $\mathbb{R}^n$ is free.
  - The rank of the family must be equal to the number of vectors.
- Solve a system of linear equations with a second member
  - Put the system in a reduced echelon form.
- Calculate the inverse of a square matrix.
  - It is equivalent to the resolution of the linear system